Quantized Harmonic Oscillators & the Electromagnetic Field:

$\boldsymbol{E}$&$\boldsymbol{B}$ radiation fields in an empty cavity with conducting walls:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B}$$
$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c}\partial_t \boldsymbol{E}$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

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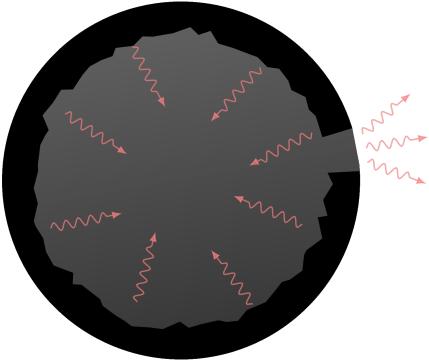
wave equation:

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\partial_t^2 \boldsymbol{E} = 0$$

Idea: Mode-function representation:

$$\boldsymbol{E}(\boldsymbol{x}, t) = \sum_m f_m(t)\boldsymbol{u}_m(\boldsymbol{x})$$



Ecce: Mode functions follow from cavity geometry as solutions to:

$$\nabla^2 \boldsymbol{u}_m(\boldsymbol{x}) = -k_m^2 \boldsymbol{u}_m(\boldsymbol{x})$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_m(\boldsymbol{x}) = 0$$
$$\hat{\boldsymbol{n}} \times \boldsymbol{u}_m(\boldsymbol{x}) = 0 \quad \text{(on cavity surface)}$$

(see cavity example below)

$\boldsymbol{E}$ & $\boldsymbol{B}$ as a collection of oscillators:

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separate, mode-specific wave equation:

$$\frac{\partial^2 f_m(t)}{\partial t^2} + c^2 k_m^2 \, f_m(t) = 0$$

total field energy:

$$H_{EB} = (8\pi)^{-1} \int_{\text{cavity}} (\boldsymbol{E}^2 + \boldsymbol{B}^2) \, \mathrm{d}^3 x$$

$$= (8\pi)^{-1} \sum_m (f_m^2 + k_m^2 \, h_m^2)$$

total energy of an infinite set of harmonic oscillators:

$$H_{\text{OSC}} = \sum_m \frac{1}{2} (P_m^2 + \omega_m^2 \, Q_m^2))$$

$\Rightarrow$ Identification: $Q_m \leftrightarrow \frac{f_m}{2\omega_m \sqrt{\pi}}$

Each *basis-field* configuration (mode) is formally **equivalent** to a classical, 1-dimensional oscillator with a characteristic frequency.

Quantization of a set of oscillators = Quantization of the ***E***&***B*** field

Transformation to occupation-number coordinates:

(position) $$Q_m = \sqrt{\frac{\hbar}{2\omega_m}}\,(a_m^\dagger + a_m)$$

(momentum) $$P_m = \frac{\mathrm{d}Q_m}{\mathrm{d}t} = i\sqrt{\frac{\hbar\omega_m}{2}}\,(a_m^\dagger - a_m)$$

$$\Rightarrow \quad H = \sum_m \hbar\,\omega_m \left(a_m^\dagger a_m + \frac{1}{2}\right)$$

Canonical quantization

$$[Q_m, P_n] = i\,\delta_{mn}\,\hbar \quad \Rightarrow \quad \left[a_m^\dagger, a_n\right] = \delta_{mn}\,\hbar$$

obtains the *electric* $$\boldsymbol{E}(\boldsymbol{x}, t) = \sum_m \sqrt{2\pi\hbar\omega_m}\left\{a_m^\dagger(t) + a_m(t)\right\}\boldsymbol{u}_m(\boldsymbol{x})$$

and *magnetic field operator* $$\boldsymbol{B}(\boldsymbol{x}, t) = \sum_m ic\sqrt{\tfrac{2\pi\hbar}{\omega_m}}\left\{a_m^\dagger(t) - a_m(t)\right\}\boldsymbol{\nabla}\times\boldsymbol{u}_m(\boldsymbol{x})$$

real effect of the ∞ zero-point energy: $\frac{2}{\text{volume}} \sum_{\boldsymbol{k}} \frac{\hbar\omega_k}{2}$ $\rightarrow$ Casimir force

E&B cavity energy with inserted wall:

$$\Delta H = H_R + H_{L-R} - H_L$$

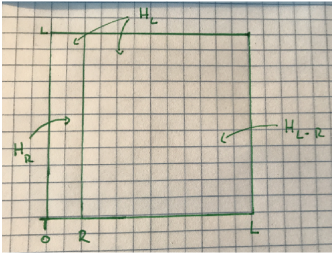
Large **vol**umes: $\frac{2}{\text{vol}} \sum_{\boldsymbol{k}} \frac{\hbar|\boldsymbol{k}|c}{2} \rightarrow \frac{2}{8\pi^3} \int_k \frac{\hbar|\boldsymbol{k}|c}{2} 4\pi k^2 \, \mathrm{d}k$

$\downarrow$

$$H_{V_l} = \frac{2V_l}{\pi^3} \int \int \int_0^\infty \frac{\hbar|\boldsymbol{k}|c}{2} \, \mathrm{d}k_x \, \mathrm{d}k_y \, \mathrm{d}k_z$$

but:

$$H_R = \sum_{n=0}^{\infty} 2 \frac{2L^2}{\pi^2} \int \int_0^\infty \frac{\hbar|\boldsymbol{k}|c}{2} \, \mathrm{d}k_y \, \mathrm{d}k_z$$



Ecce: For a rectangular cavity

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix} A_1 \cos(k_1 x) \sin(k_2 y) \sin(k_3 z) \\ A_2 \sin(k_1 x) \cos(k_2 y) \sin(k_3 z) \\ A_3 \sin(k_1 x) \sin(k_2 y) \cos(k_3 z) \end{pmatrix}$$

with $k_i = \frac{n_i \pi}{L_i}$ and

$$\omega_k = c\sqrt{k_1^2 + k_2^2 + k_3^2}$$

Physical significance:
cavity walls are conducting
only for certain frequencies!

renormalization and the Casimir force

cut off divergent $\sum$'s and $\int$'s : $\int_0^\infty \mathrm{d}k \rightarrow \int_0^\infty f\left(\frac{k}{k_c}\right) \mathrm{d}k$ with $f\left(\frac{k}{k_c}\right) \rightarrow \begin{cases} 1 & \text{for} \quad k \ll k_c \\ 0 & \text{for} \quad k \gg k_c \end{cases}$

$$\downarrow$$

$$\Delta H = \frac{\hbar c L^2 \pi^4}{4\pi^2 R^3} \left\{ \sum_{n=0}^{\infty} F(n) - \int_0^\infty F(n)\, \mathrm{d}n \right\}$$

$$= \ldots = -\hbar c \frac{\pi^2}{720} \frac{L^2}{R^3}$$

And the Casimir force per unit area of the plates is:

$$F_{\text{Casimir}} = -\frac{1}{L^2} \frac{\partial \Delta H}{\partial R} = -\frac{\hbar c}{240} \frac{\pi^2}{R^4}$$

Ponderables:

- Derive the wave equation from Maxwell's equations in free space.
- Mode expand the magnetic field and obtain a mode-specific equation for the coefficients $h_m(t)$.
- Above, we identified the electric mode coefficients f_m with coordinates via

$$Q_m \leftrightarrow \frac{f_m}{2\omega_m\sqrt{\pi}} \quad .$$

  With the alternative identification $Q_m \leftrightarrow \frac{h_m}{2c\sqrt{\pi}}$ derive the expansion of the electric field in terms of $a_m^\dagger$ and a_m.